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Cyclic Theory and the Bivariant Chern-Connes Character

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Abstract

We give a survey of cyclic homology/cohomology theory including a detailed discussion of cyclic theories for various classes of topological algebras. We show how to associate cyclic classes with Fredholm modules and K-theory classes and how to construct a completely general bivariant Chern-Connes character from bivariant K-theory to bivariant cyclic theory.

1 Introduction

The two fundamental "machines" of non-commutative geometry are cyclic homology and (bivariant) topological K-theory. In the present notes we describe these two theories and their connections. Cyclic theory can be viewed as a far reaching generalization of the classical de Rham cohomology, while bivariant K-theory includes the topological K-theory of Atiyah-Hirzebruch as a very special case.

The classical commutative theories can be extended to a striking amount of generality. It is important to note however that the new theories are by no means based simply on generalizations of the existing classical constructions. In fact, the constructions are quite different and give, in the commutative case, a new approach and an unexpected interpretation of the well-known classical theories. One aspect is that some of the properties of the two theories become visible only in the non-commutative category. For instance both theories have certain universality properties in this setting.

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Bivariant K-theory has first been defined and developed by Kasparov on the category of C^*-algebras (possibly with the action of a locally compact group) thereby unifying and decisively extending previous work by Atiyah-Hirzebruch, Brown-Douglas-Fillmore and others. Kasparov also applied his bivariant theory to obtain striking positive results on the Novikov conjecture. Very recently, it was discovered that in fact, bivariant topological K-theories can be defined on a wide variety of topological algebras ranging from discrete algebras and very general locally convex algebras to e.g. Banach algebras or C^*-algebras (possibly equipped with a group action). If E is the covariant functor from such a category of algebras given by topological K-theory or also by periodic cyclic homology, then it satisfies the following three fundamental properties:

(E1) E is diffeotopy invariant, i.e., the evaluation map ev_t in any point $t \in [0,1]$ induces an isomorphism $E(ev_t) : E(\mathfrak{A}[0,1]) \to E(\mathfrak{A})$ for any $\mathfrak{A}$ in C. Here $\mathfrak{A}[0,1]$ denotes the algebra of $\mathfrak{A}$-valued C^∞-functions on $[0,1]$.

(E2) E is stable, i.e., the canonical inclusion $\iota : \mathfrak{A} \to \mathfrak{K}\hat{\otimes}\mathfrak{A}$, where $\mathfrak{K}$ denotes the algebra of infinite $\mathbb{N} \times \mathbb{N}$-matrices with rapidly decreasing coefficients, induces an isomorphism $E(\iota)$ for any $\mathfrak{A}$ in C.

(E3) E is half-exact, i.e., each extension $0 \to \mathfrak{J} \to \mathfrak{A} \to \mathfrak{B} \to 0$ in C admitting a continuous linear splitting induces a short exact sequence $E(\mathfrak{J}) \to E(\mathfrak{A}) \to E(\mathfrak{B})$

"Diffeotopy", i.e., differentiable homotopy is used in (E1) for technical reasons in connection with the homotopy invariance properties of cyclic homology. For K-theory, it could also be replaced by ordinary continuous homotopy.

It turns out that the bivariant K-functor is the universal functor from the given category C of algebras into an additive category D (i.e., the morphism sets $D(\mathfrak{A},\mathfrak{B})$ are abelian groups) satisfying these three properties **??**.

Cyclic theory is a homology theory that has been developed, starting from K-theory, independently by Connes and Tsygan. Connes' construction was in fact directly motivated by Kasparov's formalism for bivariant K-theory and in particular for K-homology. A crucial role is played by so-called Fredholm modules or spectral triples. Also Tsygan's work is closely related to K-theory, [**?**]. In fact, in his approach, the new theory was originally called "additive K-theory" and he pointed out that it is an additive version of Quillen's definition of algebraic K-theory. It was immediately realized that cyclic homology has close connections with de Rham theory, Lie algebra homology, group cohomology and index theorems.

The theory with the really good properties is the periodic theory introduced by Connes. Periodic cyclic homology HP_* satisfies the three properties (E1), (E2), (E3), [**?**], [**?**], [**?**], [**?**], [**?**]. Combining this fact with the universality property of bivariant K-theory leads to a multiplicative transformation (the bivariant Chern-Connes character) from bivariant K-theory to bivariant periodic cyclic theory. This

transformation is a vast generalization of the classical Chern character in differential geometry.

The principal aim of this volume is to give an account of cyclic theory. Here, everything works in parallel algebraically as well as for locally convex algebras (to name just two examples think of the algebra of C^∞-functions on a smooth manifold or of algebras of pseudodifferential operators). Cyclic theory can be introduced using rather different complexes, each one of them having its own special virtues. Specifically, we will use the following complexes or bicomplexes

- The cyclic bicomplex with various realizations:

$$\begin{aligned} CC^n(A) &\quad \text{with boundary operators} \quad b, b', Q, 1-\lambda \\ \overline{CC}^n(\widetilde{A}) &\quad \text{with boundary operator} \quad B-b \\ \Omega(A) &\quad \text{with boundary operator} \quad B-b \end{aligned}$$

  The cyclic bicomplex is well suited for the periodic theory as well as for the $\mathbb{Z}$-graded ordinary theory and for the connections between both.

- The Connes complex C^n_λ
  It has the advantage, that concrete *finite-dimensional* cocycles often arise naturally as elements of C^n_λ. The connection with Hochschild cohomology also fits naturally into this picture.

- The X-complex of any complete quasi-free extension of the given algebra A. This complex is very useful for a conceptual explanation of the properties of periodic theory, in particular for proving excision, for the connections with topological K-theory and the bivariant Chern-Connes character. It also is the natural framework for all infinite-dimensional versions of cyclic homology (analytic and entire as well as asymptotic and local theory).

In the following sections we discuss the basic properties of cyclic theory. We note however, that it is quite difficult to be exhaustive and we don't even try to give a complete account of all aspects of cyclic theory. For instance, an important notion which we don't treat is the one of a cyclic object. But also other important aspects have to be omitted. We focus on those notions and results which, we think, are most relevant for non-commutative geometry, including homotopy invariance, Morita invariance, excision but also explicit formulas for the Chern character associated to idempotents, invertibles, Fredholm modules etc..

After this we turn to a description of bivariant K-theory and to the construction of the bivariant Chern-Connes character, which generalizes the Chern character for idempotents, invertibles or Fredholm modules mentioned before.

As we pointed out already above, cyclic homology and bivariant K-theory can be defined on different categories of algebras - purely algebraically for algebras (over $\mathbb{R}$

or $\mathbb{C}$) or on categories of topological algebras like locally convex algebras, Banach algebras or C*-algebras. There are different variants of the two theories which are adapted to the different categories. In this text we treat cyclic theory in the purely algebraic case (however restricting to algebras over a field of characteristic 0) on the one hand. Concerning cyclic theory for topological algebras on the other hand, we have to make a choice. For the classical ("finite-dimensional") cyclic theories we concentrate on the category of what we call m-algebras. These are particularly nice locally convex algebras (projective limits of Banach algebras). Their advantage is that, both, ordinary cylic homology and bivariant topological K-theory make perfect sense and that the bivariant Chern-Connes-character can be constructed nicely on this category. It is important to note however that the restriction to the category of m-algebras is for convenience mainly.
Cyclic theory as well as bivariant K-theory can also be treated on many other categories of algebras. In particular, for the categories of Banach algebras or C^*-algebras there are special variants of cyclic theory, namely the entire and the local cyclic theory, which are especially designed for these categories. An interesting new feature here is the existence of *infinite-dimensional* cohomology classes. These theories, as well as a bivariant character from Kasparov's KK-theory to the local cyclic theory, are discussed in sections 22 and 23 which have been written by Ralf Meyer and Michael Puschnigg.

The text starts with a collection of examples of algebras, locally convex algebras and certain extensions of algebras. These serve as a reference for later and may be omitted at a first reading.

A copy of the complete article (68 pages) can be obtained as a SFB-preprint from our homepage at wwwmath.uni-muenster.de